QPE : Quantum Phase Estimation

JM.Torres IBM Quantum France

17 novembre 2021

Agenda :

- Quantum Fourrier Transform
- Quantum Phase Estimation
- Qiskit tiny example
- Quizz

Ref : Youtube/qiskit/7. Shor's Algorithm I : Understanding Quantum Fourier Transform, Quantum Phase Estimation - Part 1.
at : https ://www.youtube.com/watch ?v=mAHC1dWKNYE&list=PLOFEBzvs-VvrXTMy5Y2IqmSaUjfnhvBHR&index=8

Motivation

For a unitary matrix U ($U^\dagger U = UU^\dagger = I, |det(U)| = 1$)

With $|\psi\rangle$ an eigenvector for U, the corresponding eigenvalue λ_ψ we have :

$$U|\psi\rangle = \lambda_\psi |\psi\rangle$$

and

$$1 = \langle\psi|\psi\rangle = \langle\psi| U^\dagger U |\psi\rangle = \langle\psi| \lambda_\psi^* \lambda_\psi |\psi\rangle = \lambda_\psi^* \lambda_\psi \langle\psi|\psi\rangle = |\lambda_\psi|^2$$

With $|\lambda_\psi|^2 = 1$, there is a θ_ψ such that $\lambda_\psi = e^{i\theta_\psi}$

Hamiltonian operators are unitary operators, having access to θ_ψ for a given eigenstate ψ is usefull in simulation.

This is Quantum Phase Estimation : given U and and eigenvector $|\psi\rangle$ we can extract θ_ψ.

Quantum Fourier Transform

For QPE, we will need QFT. QFT is really a spanning set change, from computational basis to phases space (similar to classical FT is transposing a signal into a frequency space). For one qubit : $\{|0\rangle, |1\rangle\} \rightarrow \{|+\rangle, |-\rangle\}$

Example with 5 qubits, $|43\rangle = |101011\rangle$:

Building the QFT circuit

For n qubits, their are 2^n basis states. We will also use $N = 2^n$

For a state $|x\rangle$, let us define $|\tilde{x}\rangle$ the QFT of $|x\rangle$:

$$|\tilde{x}\rangle := Q\hat{F}T\,|x\rangle := \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2i\pi x.y}{N}}\,|y\rangle$$

For one qubit, N $= 2$:

$$\left|\tilde{0}\right\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{\frac{2i\pi 0.y}{2}}\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H\,|0\rangle$$

$$\left|\tilde{1}\right\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{\frac{2i\pi 1.y}{2}}\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = H\,|1\rangle$$

QFT on 2 qubits

$$\left|\tilde{00}\right\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{3} e^{\frac{2i\pi 0.y}{4}} |y\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$\left|\tilde{01}\right\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{3} e^{\frac{2i\pi 1.y}{4}} |y\rangle = \frac{1}{2}(|00\rangle + e^{\frac{2i\pi}{4}} |01\rangle + e^{\frac{2i\pi 2}{4}} |10\rangle + e^{\frac{2i\pi 3}{4}} |11\rangle)$$

$$= \frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle)$$

$$\left|\tilde{10}\right\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{3} e^{\frac{2i\pi 2.y}{4}} |y\rangle = \frac{1}{2}(|00\rangle + e^{\frac{2i\pi 2}{4}} |01\rangle + e^{\frac{2i\pi 2.2}{4}} |10\rangle + e^{\frac{2i\pi 3.2}{4}} |11\rangle)$$

$$= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$\left|\tilde{11}\right\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{3} e^{\frac{2i\pi 3.y}{4}} |y\rangle = \frac{1}{2}(|00\rangle + e^{\frac{2i\pi 3}{4}} |01\rangle + e^{\frac{2i\pi 3.2}{4}} |10\rangle + e^{\frac{2i\pi 3.3}{4}} |11\rangle)$$

$$= \frac{1}{2}(|00\rangle + e^{\frac{3i\pi}{2}} |01\rangle + e^{3i\pi} |10\rangle + e^{\frac{9i\pi}{2}} |11\rangle) = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)$$

Leveraging binary notation of y

Back to the n qubits formula.
With $y = [y_1 y_2 ... y_n]$, this is :

$$y = 2^{n-1} y_1 + 2^{n-2} y_2 ... + 2^{n-n} y_n = \sum_{k=1}^{n} 2^{n-k} y_k = \sum_{k=1}^{n} 2^n \frac{y_k}{2^k}$$

Also : $|y\rangle = |y_1\rangle \otimes |y_2\rangle \otimes ... \otimes |y_n\rangle$

and notice :

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} |y\rangle = \frac{1}{\sqrt{N}} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} ... \sum_{y_n=0}^{1} |y_1\rangle \otimes |y_2\rangle \otimes ... \otimes |y_n\rangle$$

Alltogether :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2i\pi x.y}{N}} |y\rangle = \frac{1}{\sqrt{N}} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} ... \sum_{y_n=0}^{1} e^{\frac{2i\pi x}{N} \sum_{k-1}^{n} 2^n \frac{y_k}{2^k}} |y_1\rangle \otimes |y_2\rangle \otimes ... \otimes |y_n\rangle$$

going to next page using $\frac{N}{2^n} = 1$ and $e^{a+b} = e^a e^b$

Now :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} ... \sum_{y_n=0}^{1} \prod_{k=1}^{n} e^{\frac{2i\pi x.y_k}{2^k}} |y_k\rangle$$

swap $\prod$ and $\sum$ and develop :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \sum_{j=0}^{1} e^{\frac{2i\pi x.j}{2^k}} |j\rangle$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}}(|0\rangle + e^{\frac{2i\pi x}{2^1}} |1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^2}} |1\rangle) \otimes ... \otimes (|0\rangle + e^{\frac{2i\pi x}{2^n}} |1\rangle)$$

We went from :

$$|x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_n\rangle$$

To :

$$\frac{1}{\sqrt{N}}(|0\rangle + e^{2i\pi \frac{x}{2^1}} |1\rangle) \otimes (|0\rangle + e^{2i\pi \frac{x}{2^2}} |1\rangle) \otimes ... \otimes (|0\rangle + e^{2i\pi \frac{x}{2^n}} |1\rangle)$$

Each qubit went from state $|x_k\rangle$ to $|0\rangle + e^{2i\pi \frac{x}{2^2}} |1\rangle$

almost there...

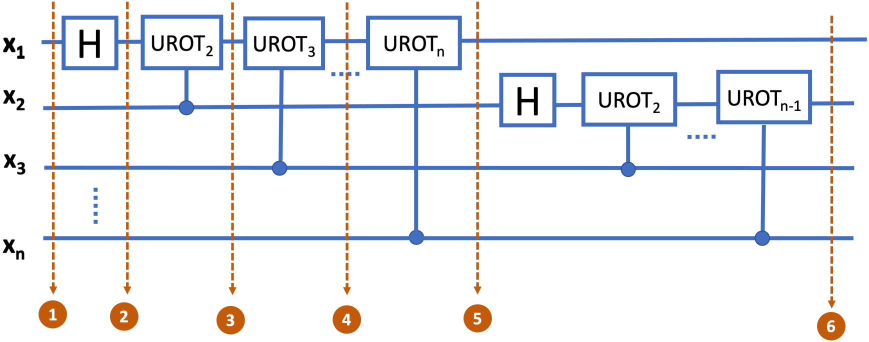
Remember :

$$H|x_k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi x_k}|1\rangle)$$

And let's define

$$UROT_j|x_k\rangle = e^{\frac{2i\pi}{2^j}x_k}|x_k\rangle \ ; \ UROT_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2i\pi}{2^j}} \end{pmatrix}$$

Building the QFT circuit



Let's see what happens step by step

Building the QFT circuit

- Step 1 : $|x_1 x_2 ... x_n\rangle$
- Step 2 : $[|0\rangle + e^{\frac{2i\pi}{2^1}x_1}|1\rangle] \otimes |x_2 x_3 ... x_n\rangle$
- Step 3 : $[|0\rangle + e^{\frac{2i\pi}{2^2}x_2} e^{\frac{2i\pi}{2^1}x_1}|1\rangle] \otimes |x_2 x_3 ... x_n\rangle$
- Step 4 : $[|0\rangle + e^{\frac{2i\pi}{2^3}x_3} e^{\frac{2i\pi}{2^2}x_2} e^{\frac{2i\pi}{2^1}x_1}|1\rangle] \otimes |x_2 x_3 ... x_n\rangle$
- Step 5 : $[|0\rangle + e^{\frac{2i\pi}{2^n}x_n} ... e^{\frac{2i\pi}{2^2}x_2} e^{\frac{2i\pi}{2^1}x_1}|1\rangle] \otimes |x_2 x_3 ... x_n\rangle$

$$= [|0\rangle + e^{(\frac{2i\pi}{2^n}x_n + \frac{2i\pi}{2^{n-1}}x_{n-1} + ... + \frac{2i\pi}{2^2}x_2 + \frac{2i\pi}{2^1}x_1)}|1\rangle] \otimes |x_2 x_3 ... x_n\rangle$$

$$e^{(\frac{2i\pi}{2^n}x_n + \frac{2i\pi}{2^{n-1}}x_{n-1} + ... + \frac{2i\pi}{2^2}x_2 + \frac{2i\pi}{2^1}x_1)} = e^{2i\pi(\frac{x_n}{2^n} + \frac{x_{n-1}}{2^{n-1}} + ... + \frac{x_1}{2^1})}$$

and

$$x = 2^0 x_n + 2^1 x_{n-1} + ... + 2^{n-1} x_1 \Leftrightarrow \frac{x}{2^n} = \frac{x_n}{2^n} + \frac{x_{n-1}}{2^{n-1}} + ... + \frac{x_1}{2^1}$$

so Step 5 is : $(|0\rangle + e^{\frac{2i\pi x}{2^n}}|1\rangle) \otimes |x_2 x_3 ... x_n\rangle$, then :

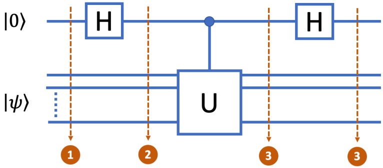
- Step 6 : $(|0\rangle + e^{\frac{2i\pi x}{2^n}}|1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^{n-1}}}|1\rangle)\,|x_3 x_4 ... x_n\rangle$
- Step '7' : $(|0\rangle + e^{\frac{2i\pi x}{2^n}}|1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^{n-1}}}|1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^{n-2}}}|1\rangle)\,|x_4 x_5 ... x_n\rangle$
- Step 'final' : $(|0\rangle + e^{\frac{2i\pi x}{2^n}}|1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^{n-1}}}|1\rangle) \otimes ... \otimes (|0\rangle + e^{\frac{2i\pi x}{2^1}}|1\rangle)$

Now to the QPE !

Let's recall that a unitary matrix has eigenvectors ψ (that form an orthonormal basis) with eigenvalues of the form $e^{i\theta_\psi}$:

$$U\left|\psi\right\rangle = e^{i\theta_\psi}\left|\psi\right\rangle$$

We want to extract θ_ψ given the ability to prepare $\left|\psi\right\rangle$, QPE trick :



Again, let's see what happens step by step

Building the QFT circuit

- Step 1 : $|0\rangle\,|\psi\rangle$
- Step 2 : $\frac{1}{\sqrt{2}}(|0\rangle\,|\psi\rangle + |1\rangle\,|\psi\rangle)$
- Step 3 : $\frac{1}{\sqrt{2}}(|0\rangle\,|\psi\rangle + |1\rangle\,U|\psi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle\,|\psi\rangle + |1\rangle\,e^{i\theta_\psi}\,|\psi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle\,|\psi\rangle + e^{i\theta_\psi}\,|1\rangle\,|\psi\rangle)$
- Step 4 : $\frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|\psi\rangle + e^{i\theta_\psi}\,\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\;|\psi\rangle)$

$$= \frac{1}{2}(|0\rangle + |1\rangle)\,|\psi\rangle + e^{i\theta_\psi}(|0\rangle - |1\rangle)\;|\psi\rangle)$$

$$= \frac{1}{2}(|0\rangle\,(1 + e^{i\theta_\psi}) + |1\rangle\,(1 - e^{i\theta_\psi}))\,|\psi\rangle$$

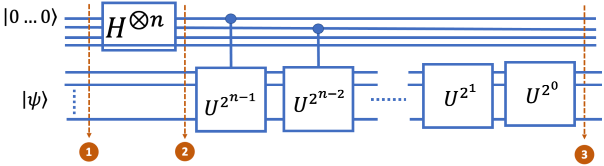
Measuring qubit 0 : $P(|0\rangle) = |\frac{1+e^{i\theta_\psi}}{2}|^2 = \cos^2(\frac{\theta_\psi}{2})$; $P(|1\rangle) = |\frac{1-e^{i\theta_\psi}}{2}|^2 = \sin^2(\frac{\theta_\psi}{2})$
Not easy to measure θ_ψ for relatively small angles :

For $\theta_\psi = 1^\circ$: $P(|0\rangle) = 0.9999$ and $P(|1\rangle) = 7.615 \times 10^{-5}$

For $\theta_\psi = 10^\circ$: $P(|0\rangle) = 0.9924$ and $P(|1\rangle) = 7.596 \times 10^{-4}$
Measuring the phase on one qubit yields low precision.

QPE circuit

Better solution = use multiple qubits to extract the phase :



Again, let's see what happens step by step

Building the QPE circuit...

- Step 1 : $|0\rangle^{\otimes n}|\psi\rangle$
- Step 2 : $\frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle)^{\otimes n}|\psi\rangle$

  Note that $U^{2^k}|\psi\rangle = U^{2^k-1}U|\psi\rangle = U^{2^k-1}e^{i\theta_\psi}|\psi\rangle = ... = e^{i\theta_\psi 2^k}|\psi\rangle$

  ...
- Step 3 : $\frac{1}{\sqrt{2^n}}(|0\rangle + e^{i\theta_\psi 2^{n-1}}|1\rangle) \otimes (|0\rangle + e^{i\theta_\psi 2^{n-2}}|1\rangle) \otimes ... \otimes (|0\rangle + e^{i\theta_\psi 2^0}|1\rangle)$

  remember QFT form :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}}(|0\rangle + e^{\frac{2i\pi x}{2^1}}|1\rangle) \otimes (|0\rangle + e^{\frac{2i\pi x}{2^2}}|1\rangle) \otimes ... \otimes (|0\rangle + e^{\frac{2i\pi x}{2^n}}|1\rangle)$$

Same form with changing θ_ψ to $2\pi\frac{x}{2^n}$ so we will get access to θ_ψ when applying QFT^{-1} on ancilla register.

Measurement should yield $\frac{2^n}{2\pi}\theta_\psi$